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Probability and Heisenberg Uncertainty of He^+ at Quantum Numbers n≤3

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ABSTRACT

The purpose of this study is to analyze the probability and uncertainty of electron linear momentum in $He^+$with the Heisenberg uncertainty approach. Measurement of the position and momentum of atomic electrons is probabilistic. The probability and uncertainty of electron linear momentum are analysed analytically, and simulations of hydrogenic atoms' normalized radial wave function are performed. He^+ they can be viewed as hydrogenic atoms with only one electron orbital. The probability and uncertainty of electron linear momentum in He^+decrease with increasing values of the principal quantum number n ≤ 3. While the uncertainty of the electron position is increasing. The results of this study are in accordance with the characteristics of position and linear momentum that are not commutable. The increase in the value of the main quantum number means that the electron's position against this is getting farther, and the speed in the orbital is getting smaller.

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I. Introduction

Based on the consideration of the symmetry of nature, in 1924, de Broglie hypothesized that if waves can be particles, then particles should also be viewed as waves. As a result of this hypothesis about particle-wave dualism, a new theory known as quantum mechanics was developed [1]. In quantum physics, the physical quantities resulting from measurement are probabilistic. Uncertainty is a fundamental property associated with the measurement of physical quantities. The Heisenberg uncertainty principle adds further complexity to the understanding of subatomic behaviour. This principle asserts that fundamental limits exist to measure a particle's position and momentum simultaneously. The higher the accuracy of a particle's position measurement, the greater the uncertainty in its momentum, and vice versa [2]. According to Born's interpretation, the absolute value of the square of the wave function, $|\psi_{r,t}|^2$ indicates the probability density of the electron in the nucleus's vicinity. In quantum mechanics, the position of an electron can only be described statistically [3]. $\psi_{(r)}$ is the Schrodinger wave function is the solution of the time-free (steady-state) non-relativistic Schrodinger equation that can provide comprehensive quantum system information [4].

Probability is the chance of finding an electron in an atom or ion. Analytically, the probability of getting an electron in a hydrogenic atom is

$$P = \int_{-\infty}^{\infty} |\Psi_{r,t}|^2 \, dV \; [5] \qquad (1)$$

This calculation results in the electron radial probability density/distribution, which is a number that provides information about the relative probability of finding an electron at a particular point in space [5]. The radial probability distribution depends on the main quantum number (n) and azimuthal (l), determining the chance of finding an electron at a certain distance from the nucleus, with nodes in higher-energy orbitals and probability peaks around the Bohr radius [6]. Ref's research supports this [7] that the larger the values of n and l, the more vertices and the more complex the orbital shape, so the radial probability distribution helps to understand the position of the electron and its interaction with the atomic nucleus. The average position values in hydrogenic atoms for the ground state n = 1 show that electrons are most likely to be at an average distance equal to the Bohr radius. In addition, the average of momentum values shows a quantized electron momentum distribution, increasing at higher energy levels. This fact aligns with quantum mechanical theory, which states that neither the position nor the momentum of electrons in the hydrogen atom can be determined with certainty but only in a probability distribution [8].

Based on the principle of wave-particle dualism, uncertainty is not a limitation of measuring instruments but a fundamental characteristic of quantum reality itself [9]. The position of an electron in an atom is probabilistic because it changes as a function of its position. Electron position uncertainty Δr and electron momentum uncertainty Δp in atoms cacn be solved analytically and numerically using the equation (2) [9].

$$\Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} \text{ and } \quad \Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} \tag{2}$$

This is supported by the research of Ref [10] which shows that the uncertainty of position and momentum in hydrogenic atoms can be analyzed using the method of average of the square value of position and momentum. Ref [11] states that the position of electrons in an atom cannot be determined with certainty, which can be determined by the probability of finding electrons as a function of distance from the atomic nucleus. An increase in the principal quantum number and orbitals leads to a decrease in the probability. However, the greater the interval value of the particle position r measured from the atomic nucleus, the greater the probability value because this probability value is directly proportional to r^2.

He^+ are ionized helium atoms that release an electron so that they only have one orbital electron. The utilization of He^+in everyday life includes cooling when in a liquid state [12], as a semiconductor magnet such as in Magnetic Resonance Imaging (MRI) and Nuclear Magnetic Resonance (NMR) [13]. In the industrial field, He^+can help increase the spotting rate of beryllium materials to make thin films in various industries, which can later be used to make various things, such as making semiconductors, hard drives, and solar panels [14]. Therefore, it is necessary to review the basic behaviour of helium ion electrons, including Heisenberg's probability and uncertainty in the position space wave function.

II. Theory

Radial Wave Function of Hydrogenic Atoms

Solving problems in hydrogenic atoms can be done by solving the Schrodinger equation in a spherical coordinate system. By using the potential energy value,

$$V = -\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}\nabla^2\Psi_r + \frac{2m}{\hbar^2}(E-V)\Psi_r = 0 \tag{3}$$

the Schrodinger equation of the hydrogenic atom is written as:

$$\nabla^2\Psi_r + \frac{2m}{\hbar^2}(E-V)\Psi_r = 0 \tag{4}$$

The radial Schrodinger equation and the angular Schrodinger equation are obtained using the variable separation method. The solution of the radial part of the Schrodinger equation is the radial wave function. The normalized radial wave function of a hydrogenic atom is obtained by applying the Laquerre function and the associated Laguerre function,

$$R_{nl}(r) = N_{nl}\, e^{-\frac{zr}{na_0}} \left(\frac{2zr}{na_0}\right)^l L_{n+1}^{2l+1}\left(\frac{2zr}{na_0}\right) \tag{5}$$

with $N_{nl} = -\left(\frac{2z}{na_0}\right)^{3/2}\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}}$ [15]. The radial wave functions of hydrogenic atoms at n ≤ 3 are shown in Table 1.

Table 1. Normalized Radial Wave Function of Hydrogenic Atom at n ≤ 3

n	*l*	Orbital	R_{nl}(r)
1	0	1*s*	$2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$
2	0	2*s*	$\frac{1}{2\sqrt{2}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}(2-\frac{Zr}{a_0})e^{-Zr/a_0}$
2	1	2*p*	$\frac{1}{4\sqrt{6}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\frac{Zr}{a_0}e^{-Zr/a_0}$
3	0	3*s*	$\frac{1}{9\sqrt{3}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\left(6-\frac{4Zr}{a_0}+\frac{4Z^2}{9a_0^2}\right)e^{-Zr/3a_0}$
3	1	3*p*	$\frac{1}{27\sqrt{6}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\left(4-\frac{2Zr}{3a_0}\right)\frac{2Zr}{a_0}e^{-Zr/3a_0}$
3	2	3*d*	$\frac{1}{81\sqrt{30}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}}\frac{4Z^2r^2}{a_0^2}e^{-Zr/3a_0}$

Probability and Expectation of Electron Position

The probability equation for finding an electron in a hydrogenic atom is expressed in equation (1). Since probability is a function of position, with the separation of variables method, the probability equation can also be expressed as:

$$P_{nl}(r) = \int_{-\infty}^{\infty} r^2|R_{nl}(r)|^2\, dr \tag{6}$$

$|R_{nl}(r)|$ is the normalized radial wave function [11]. The position of electrons in the atom is probabilistic, so the average position of electrons in the atom is called the expectation value.

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The average value of the position or expectation of the electron position in the atom is formulated as [15]:

$$\langle r\rangle = \int_{-\infty}^{\infty} r|\Psi(r,t)|^2\, dv = \int_{-\infty}^{\infty} \Psi^*_{(r,t)} r\, \Psi_{(r,t)}\, dV \quad (7)$$

In spherical coordinates, the expectation value can also be formulated:

$$\langle r\rangle = \int_{-\infty}^{\infty} r^3 |R_{nl}(r)|^2\, dr \quad (8)$$

From equation (8), the electron's position expectation value is determined by the principal quantum number *n* and the azimuthal quantum number *l*. Ref [16] indicates that as the principal quantum number and azimuthal quantum number increase, the electron probability value decreases. Therefore, there is a possibility of not finding electrons at increasingly far orbital distances. Based on equation (7), the value of the square expectation of the electron position can be formulated:

$$\langle r^2\rangle = \int_{-\infty}^{\infty} \Psi^*_{(r,t)} r^2\, \Psi_{(r,t)} dV = \int_{-\infty}^{\infty} r^4\, |R_{nl}(r)|^2\, dr \quad (9)$$

Heisenberg Uncertainty of the Hydrogenic Atom

Heisenberg's uncertainty explains that it is impossible to simultaneously measure a particle's position and momentum with infinite accuracy [17]. The uncertainty value between the position and momentum operators is addressed in equation (10).

$$\Delta r \Delta p\ \geq \frac{1}{2}|\langle[\tilde{r},\tilde{p}]\rangle| \quad (10)$$

$\tilde{r} = r$ and $\tilde{p} = -i\hbar\frac{\partial}{\partial r}$ are the position operator and momentum operator, respectively. The position and momentum operators are non-commuting, $[\tilde{r},\tilde{p}] = i\hbar$. According to the definition of expected value and the commutator property, we get:

$$\langle \tilde{r},\tilde{p}\rangle = \frac{1}{2}\left|\int_{-\infty}^{\infty} \psi^*(i\,\hbar)\,\psi\, dr\,\right| \quad (11)$$

Then equation (10) can be simplified to:

$$\Delta r \Delta p\ \geq \frac{1}{2}|i\,\hbar| \text{ or} \quad (12)$$

$$\Delta r \Delta p \geq \frac{\hbar}{2} \quad (13)$$

Based on equation (13), the more Δr increases, the more the wave packet will spread (the wave nature is clearer), and the particle nature is less clear [18]. Electron position uncertainty is the standard deviation by taking the root of the variance [19]. So, the equation for the position uncertainty value is written in the equation:

$$\Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} \quad (14)$$

Based on the Heisenberg uncertainty equation (13), the momentum uncertainty can be expressed as:

$$\Delta p \geq \frac{\hbar}{2\Delta r} \quad (15)$$

Ions of He (He^+)

Helium is a chemical element in the periodic table with the symbol He and atomic number 2. It is known that this element has eight isotopes [18]. Like noble gases in general, Helium has stable, unreactive energy and a high ionization energy [19]. Positive He^+ are formed from Helium atoms in which one electron is ionized and leaves one electron. After one of the electrons is ionized, the helium atom becomes a helium ion and behaves as a hydrogenic atom [20].

He^+ are commonly observed in hot stars like the sun due to the high temperatures that cause helium atoms to move rapidly. In the sun's very hot atmosphere, these atoms collide with other atoms, releasing electrons and forming Helium ions. (He^+) After losing one electron, this ion has only one electron left, so it is hydrogenic [21].

Using the Bohr atomic model, the radius of the electron orbitals in the hydrogenic atom is $r_n = n^2\, a_0$. $n = 1, 2, 3, \cdots$ is the main quantum number, a_0 = 0.02645 nm is the radius of the helium ion in the ground state n = 1 mathematically formulated:

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{Zme^2} \quad (16)$$

and m is reduction mass of He^+ formulated:

$$m = \frac{m_e\, m_I}{m_e + m_I} \quad (17)$$

with $m_e = 9.10938215 \times 10^{-31} kg$ is a mass of electron and $m_I = 6.644\, x\, 10^{-27}\, kg$ is a helium ion nuclei mass [9].

III. Method

This research employs a non-experimental method, building upon existing theories. This research is to calculate the value of electron probability and momentum uncertainty through the Heisenberg uncertainty equation for helium ions. This study aims to analyze the relationship between the value of the main quantum number, n with the probability value, $P(r)$ and the value of momentum uncertainty, Δp through the Heisenberg uncertainty equation.

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The first step in this study is to determine the radius of the hydrogen atom in the ground state, a_0 using equation (16), namely:

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{Zme^2}$$

With reduction mass $= \frac{m_e m_I}{m_e + m_I}$. Next, find the probability of electrons in hydrogen ions at quantum numbers $n = 1, 2\ dan\ 3$ using equation (6), namely:

$$P_{nl}(r) = \int_a^b r^2 |R_{nl}(r)|^2\, dr \qquad (18)$$

Where $|R_{nl}(r)|$ s the radial function of electrons in He^+ as shown in Table 1.

The uncertainty of helium ion electron momentum, Δp using the Heisenberg uncertainty approach, goes through several stages, namely:

(i) Determine the average of the position and average of the square position of helium ion electrons at quantum numbers $n = 1, 2, \&\ 3$ using the equation:

$$\langle r\rangle = \int_{r_1}^{r_2} r^3\, |R_{nl}(r)|^2\, dr \qquad (19a)$$
$$\langle r^2\rangle = \int_{r_1}^{r_2} r^4\, |R_{nl}(r)|^2\, dr \qquad (19b)$$

(ii) Determine the position uncertainty, Δr by using equation (14), that is:

$$\Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} \qquad (20)$$

(iii) Calculate the Helium-ion electron momentum uncertainty $(n \le 3)$ using Heisenberg's Uncertainty principle, that is:

$$\Delta p = \frac{\hbar}{2\Delta r} \qquad (21)$$

The probability and uncertainty of electron position in He^+at quantum number $n \le 3$ are performed analytically and also numerically. The uncertainty of momentum is analyzed using an analytical approach.

Numerical probability calculation using Matlab R2021b simulation. The stages are Flowchart in Figure 1.

IV. Results and Discussion

The research was conducted by calculating the probability value to find electrons and the uncertainty of the position of helium ion electrons at $n \le 3$ analytically and numerically. The uncertainty of electron momentum is calculated analytically through the radial wave function in position space through the Heisenberg uncertainty equation or according to equation (13).

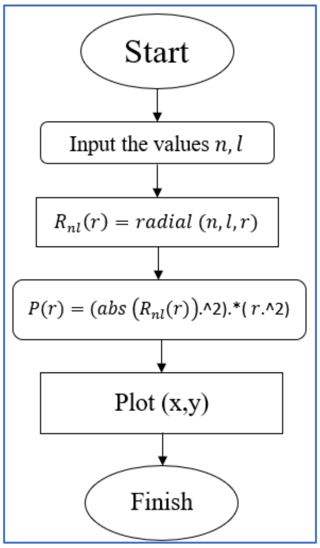

Figure 1. Numerical Probability Flowchart

The mass of $He^+$is the combined reduced mass between the mass of the nucleus and the mass of electrons as formulated in equation (17), as obtained:

$$m = \frac{(6.644\ x\ 10^{-27})(9.1094 \times 10^{-31})}{6.644\ x\ 10^{-27} + 9.1094 \times 10^{-31}}$$
$$= 9.0987\ x\ 10^{-31} kg$$

Using equation (16), In the ground state (n = 1), the helium ion has a radius of $a_0 = 2.65\ x\ 10^{-11}\ m$. The atomic number will be inversely proportional to the atomic radius because the more protons, the smaller the distance between the atomic nucleus and its outer electrons. The mass of reduced $He^+$is smaller than the rest mass of the electrons, so it can affect the atomic radius because the number of neutrons and binding forces in this will be greater with the radius of the electron trajectory getting smaller [22].

Information about electron behaviour can be known through wave functions. The radial wave function solution in Table 2 can be used to determine the electron's probability value and the Heisenberg uncertainty. The normalized radial wave function depends on the principal quantum number (n) and orbital quantum number (l).

The probability of finding an electron in a helium ion at the principal quantum number n ≤ 3 is obtained through equation (18) using integration limits a = 0 and $b = a_0, 2a_0, \cdots, 9a_0$. Analytically, the probability value at $(n, l) = (1, 0)$ with $b = a_0$ is:

$$P_{10} = \int_0^{a_0} r^2 |R_{10}|^2 dr$$
$$= \int_0^{a_0} r^2 \left| 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0} \right|^2 dr$$
$$= \frac{32}{a_0^3} \int_0^{a_0} r^2\, e^{-\frac{4r}{a_0}} dr$$

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$$= \frac{32}{a_0^3}\left[-r^2\frac{a_0}{4}e^{-\frac{4r}{a_0}} - 2r\frac{a_0^2}{16}e^{-\frac{4r}{a_0}} - 2\frac{a_0^3}{64}e^{-\frac{4r}{a_0}}\right]_0^{a_0}$$

$$= \frac{32}{a_0^3}\left\{\frac{\left(-\frac{a_0^3}{4}-\frac{a_0^3}{8}-\frac{a_0^3}{32}\right)}{e^4} - \left(-\frac{a_0^3}{32}\right)\right\}$$

$$= 32\left[\left(\frac{-13}{32(2,71828183)^4}\right)+\frac{1}{32}\right]$$

So, the probability of finding helium ion electrons in the combination of $(n,l) = (1,0)$ with $b = a_0$ is $P_{10} = 0.761896704$ or about 76.19 %. While $(n,l) = (3,2)$ obtained $P_{32} = 0.0004682$ or 0.047% in the region with a radius $r = a_0$ and $P_{32} = 0.95417769342$ or about 95.42% in the region with electron orbital radius $r = 9a_0$.

Numerical calculation of probability values using Simpson's numerical integration method through Matlab R2021b simulation. The simulation method has been validated by comparing the results of the probability and average of electrons with an average error below 0.001 on other hydrogenic atoms. The complete results of numerical calculations of the probability of finding electrons in He^+at $n \leq 3$ are presented in Table 2.

Table 2. Electron Probability

r	$n = 1$	$n = 2$		$n = 3$		
	$l = 0$	$l = 0$	$l = 1$	$l = 0$	$l = 1$	$l = 2$
a_0	0.7618966944467	0.0526530173437	0.0526530173437	0.0143532099691	0.0169244974007	0.0004682578378
$2a_0$	0.9862460322603	0.1757962500075	0.3711630648201	0.0535487650770	0.0887935335069	0.0193884510601
$3a_0$	0.9994777419716	0.5364733429661	0.7149434996834	0.1106739784034	0.1106739784026	0.1106739784026
$4a_0$	0.9999836824679	0.8144891667785	0.9003675995130	0.1171017983033	0.1336791580723	0.2879993072182
$5a_0$	0.9999995446517	0.9404806921022	0.9707473119230	0.1783256640072	0.2512581909487	0.4995391337493
$6a_0$	0.9999999885294	0.9835519435733	0.9923996093189	0.3373867626547	0.4423492915501	0.6866257224636
$7a_0$	1.0000000003491	0.9959130364256	0.9981947511508	0.5375365667322	0.6363544071582	0.8219189267092
$8a_0$	1.0000000010853	0.9990617943849	0.9995995623366	0.7143517332237	0.7874015537858	0.9066095478845
$9a_0$	1.0000000017492	0.9997973957981	0.9999158239019	0.8402197578165	0.8862276198534	0.9541776931113

The results of analytical and numerical probability calculations give the same results, namely, the probability value of electrons in $He^+$is getting smaller with the increase in the value of the main quantum number. Increasing the value of the main quantum number provides information that the radius of the electron orbital in the ion is getting bigger. In the Bohr atomic model, the radius of the electron orbit is formulated as $r_n = n^2 a_0$. The larger the orbital radius, the smaller the probability density value [1]. Based on Table 2, the smallest probability value on the quantum cross $(n,l) = (3,2)$ at the position of the electron $r = a_0$, which is 0.0004682. This result shows that electrons are very difficult to find or even not found in the orbital.

Table 2 also shows the probability value of finding the largest electron in the quantum number n = 1 when the electron position $r = a_0$. In the combination of $(n,l) = (1,0)$ the probability value of electrons in the orbit region $r = 0 - a_0$ is 76.19%, while in the region $a_0 - 4a_0$ is 23.80%, and in the region $4a_0 - 9a_0$ is 0.01%. At $(n,l) = (2,1)$, the probability value in the $r = 0 - a_0$ orbit region is 5.26%, while in the region $a_0 - 4a_0$ is 84.77% and in the region $4a_0 - 9a_0$ is 9.95%. And for quantum number $(n,l) = (3,2)$ the probability value in orbit $r = 0 - a_0$ is 0.05%, while in the region $a_0 - 4a_0$ is 28.75 % and in the region $4a_0 - 9a_0$ is 66.62%. This means that the electrons are most often in the position $r_n = n^2 a_0$ so the probability is getting bigger. The probability of finding electrons decreases as the distance from the atomic nucleus increases and approaches zero at a very long distance. Hence, the possibility of finding electrons in the region is very small. The probability of finding an electron on a hydrogenic atom, Deuterium, in momentum space, decreases with the increasing value of electron momentum. In momentum space, the value of electron momentum $p_n = n^2 p_0$ [4]. The probability meeting of electrons in molecular ions H_2^+P is getting smaller with the distance of electrons to the nucleus under symmetry conditions (if the distance between protons is close to zero, it appears that the electron distribution in the ground state is like a single atom with 2 protons in the nucleus) [11].

In representing the probability of electrons in space per unit volume can be shown through the following probability density graph in Figure 2.

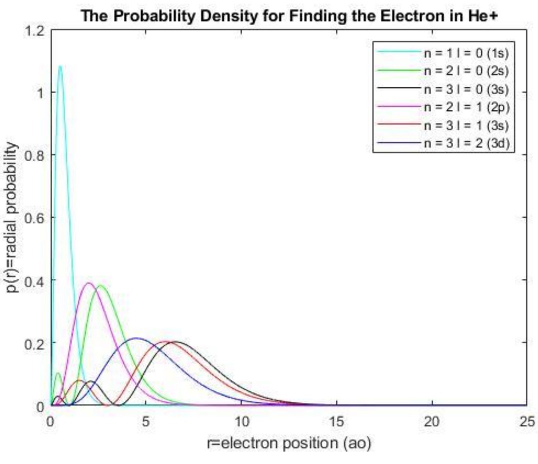

Figure 2. Probability Distribution of Helium Ion Electron

The probability distribution graph above shows the graph of the $P(r)$ function in various orbitals. The largest probability is in the orbital $r = a_0$ at quantum number *n* = 1. An increase in quantum number leads to decreased probability density peaks due to orbital expansion. The greater the value of the orbital quantum number *l*, the peak of the distribution shifts towards a smaller distance, *r*. The orbital quantum number *l* is also called the orbital angular momentum quantum number which determines the amount of orbital angular momentum and describes the shape of the orbital. Orbital *s* or *l* = 0, electrons have the simplest symmetrical spherical orbital, which indicates that electrons have the same density, while at *l* = 1 or orbital *p*, the electron density is not evenly distributed but concentrated in two regions that are divided equally and located on two opposite sides of the core located in the centre [23]. The electron probability distribution of He^+presented in Figure 2 has the same pattern as the electron probability distribution of deuterium atoms in momentum space [24] and molecular systems under the influence of the Kratzer potential [25].

The position of electrons in $He^+$is probabilistically. Therefore, the average position of helium electrons at the (n, l) is calculated using equation (19a) with specified integration limits $r_1 = 0$ dan $r_2 = a_0, 2a_0, \cdots, 9a_0$. Analytically, the expected value of the electron position at $(n, l) = (1, 0)$ with $r_2 = a_0$ is:

$$\langle r \rangle = \int_0^{a_0} r^3 |R_{10}|^2 dr$$
$$= \int_0^{a_0} r^3 \left| 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0} \right|^2 dr$$
$$= \frac{32}{a_0^3} \int_0^{a_0} r^3 e^{-4r/a_0} dr$$
$$= \frac{32}{a_0^3} \left[\left(-r^3 \frac{a_0}{4} - 3r^2 \frac{a_0^2}{16} - 6r \frac{a_0^3}{64} - 6 \frac{a_0^4}{256} \right) e^{-\frac{4r}{a_0}} \right]_0^{a_0}$$
$$= \frac{32}{a_0^3} \left[\left(\frac{-0{,}5546875 a_0^4}{2{,}71828183^4} \right) - (-0{,}0234375 a_0^4) \right]$$

So, the expected value of the position of helium ion electrons at $(n, l) = (1, 0)$ using $r_2 = a_0$ is $\langle r \rangle = 0.42489741056 a_0$. The expected value of the square of the position of helium ion electrons in the same quantum number state and integration limit is:

$$\langle r^2 \rangle = \int_0^{a_0} r^4 |R_{10}|^2 dr$$
$$= \int_0^{a_0} r^4 \left| 2\left(\frac{2}{a_0}\right)^{3/2} e^{-2r/a_0} \right|^2 dr$$
$$= \frac{32}{a_0^3} \int_0^{a_0} r^4 e^{-4r/a_0} dr$$
$$= \frac{32}{a_0^3} \left[-r^4 \frac{a_0}{4} e^{-\frac{4r}{a_0}} - 4r^3 \frac{a_0^2}{16} e^{-\frac{4r}{a_0}} - 12r^2 \frac{a_0^3}{64} e^{-\frac{4r}{a_0}} - 24r \frac{a_0^4}{256} - 24 \frac{a_0^5}{1024} e^{-\frac{4r}{a_0}} \right]_0^{a_0}$$
$$= \frac{32}{a_0^3} \left[\left(\frac{-0{,}8046875 a_0^5}{2{,}71828183^4} \right) - (-0{,}0234375 a_0^5) \right]$$

So, the expected value of the square of the position of the helium ion electron at $(n, l) = (1, 0)$ using $r_2 = a_0$ is $\langle r^2 \rangle = 0.27837229984 a_0^2$. By using the expected value of the position and the expected square of the position of the helium ion electron, the uncertainty of its electron position can be calculated using equation (20), namely:

$$\Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$$
$$= 0.3128\, a_0 = 0.08270094174 \times 10^{-10} m$$

Analytical position uncertainty in $(n, l) = (2, 1)$ with $b = 4a_0$ is $\Delta r = 1.0280\, a_0 = 0.27242031801 \times 10^{-10}\, m$. While at $(n, l) = (3, 2)$ with $b = 9a_0$ is $\Delta r = 1.940941927 a_0 = 0.513188603 \times 10^{-10} m$. In the simulation, the uncertainty of the electron position of He^+in position space at n ≤ 3 is presented in Table 3.

Table 3. Helium Ion Electron Position Uncertainty

n	l	r	Δr
1	0	a_0	$0.082700941535841 \times 10^{-10}$
		$4a_0$	$0.114429634507704 \times 10^{-10}$
		$9a_0$	$0.114489352289300 \times 10^{-10}$
2	0	a_0	$0.027735379555852 \times 10^{-10}$
		$4a_0$	$0.335375424116796 \times 10^{-10}$
		$9a_0$	$0.323050293897786 \times 10^{-10}$
	1	a_0	$0.047510614271400 \times 10^{-10}$
		$4a_0$	$0.272420318012262 \times 10^{-10}$
		$9a_0$	$0.295163339984104 \times 10^{-10}$
3	0	a_0	$0.014128940498643 \times 10^{-10}$
		$4a_0$	$0.182109467929346 \times 10^{-10}$
		$9a_0$	$0.761150495880383 \times 10^{-10}$
	1	a_0	$0.027237113711577 \times 10^{-10}$
		$4a_0$	$0.195802659687906 \times 10^{-10}$
		$9a_0$	$0.694288693276678 \times 10^{-10}$
	2	a_0	$0.004951810916410 \times 10^{-10}$
		$4a_0$	$0.383032020024593 \times 10^{-10}$
		$9a_0$	$0.513188603004719 \times 10^{-10}$

From Table 3, it can be seen that the uncertainty of the position of the helium ion electron, Δr with $r_n = n^2 a_0$ will get bigger with increasing quantum number values. Based on equation (14), the position uncertainty is proportional to the root of the difference between the squared position average and the squared position average. The main quantum number n affects the expected value of the Lithium-ion position [24].

Position and momentum operators do not commute, mathematically formulated as $[\hat{r}, \hat{p}_r] = i\hbar \neq 0$. This means that position and linear momentum cannot be measured simultaneously. Measurement of particle momentum in the *x*-axis direction can be done with uncertainty Δp_x then to measure position in the *x*−axis direction simultaneously cannot have an accuracy greater

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than $\Delta x = \frac{\hbar}{2\Delta p_x}$ [8]. The uncertainty relationship of position and momentum is expressed in the Heisenberg uncertainty equation as formulated in equation (13), namely:

$$\Delta r \cdot \Delta p \geq \frac{\hbar}{2}; \text{ with } \hbar = \frac{h}{2\pi} = 1{,}054 \times 10^{-34} Js$$
$$\text{or } \Delta p \geq \frac{\hbar}{2\Delta r}.$$

The measurement of the electron position of $He^+$is done with the uncertainties as presented in Table 3, and then the maximum accuracy of the measurement of the momentum of the helium ion simultaneously is $\Delta p = \frac{\hbar}{2\,\Delta r}$. This means that the minimum linear momentum uncertainty of the helium ion electron in its position space is $\Delta p = \frac{\hbar}{2\,\Delta r}$. Details are presented in Table 4.

Table 4. Momentum Uncertainty of Helium Ion Electrons in Position Space

n	l	r	Δp
1	0	a_0	$0.637235792257103 \times 10^{-23}$
		$4a_0$	$0.460545034743182 \times 10^{-23}$
		$9a_0$	$0.460304813908228 \times 10^{-23}$
2	0	a_0	$0.190010019130530 \times 10^{-22}$
		$4a_0$	$0.015713733389614 \times 10^{-22}$
		$9a_0$	$0.016313249359456 \times 10^{-22}$
	1	a_0	$0.110922581844460 \times 10^{-22}$
		$4a_0$	$0.019345106262459 \times 10^{-22}$
		$9a_0$	$0.017854520823229 \times 10^{-22}$
3	0	a_0	$0.372993289943167 \times 10^{-22}$
		$4a_0$	$0.028938638171436 \times 10^{-22}$
		$9a_0$	$0.006923729313090 \times 10^{-22}$
	1	a_0	$0.193485993259264 \times 10^{-22}$
		$4a_0$	$0.026914854008623 \times 10^{-22}$
		$9a_0$	$0.007590502410645 \times 10^{-22}$
	2	a_0	$0.106425711501543 \times 10^{-21}$
		$4a_0$	$0.001375864085635 \times 10^{-21}$
		$9a_0$	$0.001026912906706 \times 10^{-21}$

From Table 4, it can be seen that the uncertainty of the momentum of helium ion electrons in position space is as follows:

(1) It decreases with the further position of the electron to the nucleus;
(2) It decreases with increasing main quantum number;
(3) It decreases with increasing orbital quantum number.

With the smaller the uncertainty, the measurement accuracy of helium ion electron momentum increases.

V. Conclusion

The probability of a helium ion electron in position space depends on the distance of the electron to the atomic nucleus. The larger the main quantum number, the larger the electron orbital and the smaller the probability of finding the electron. The highest electron position probability meeting when n = 1 and the peak of the probability meeting will shift towards r getting smaller as the orbital quantum number increases. The uncertainty of the position of helium ion electrons is getting bigger as the main quantum number increases. Momentum measurement accuracy increases with increasing quantum number n.

The uncertainty of electron momentum through the Heisenberg uncertainty equation assumes that the radial wave function is a Gaussian function, so $\Delta p\ \Delta r = \hbar/2$. Generally, the wave function of physical quantities is not always Gaussian.

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Declarations

Author contribution	:	Bambang Supriadi was responsible for the overall research project and led the manuscript writing and collaboration with all authors. Zidan Afidah was responsible for data collection and data analysis. Maya Arsita reviewed the theoretical writing and accuracy. Simatun Ni'mah reviewed the research steps and edited the manuscript. Merry Khanza Kusuma Wardhany is a contact person and scriptwriter. Dimas Feri Hermansyah wrote and edited the manuscript. All authors approved the final manuscript.
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