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The role of mathematical formulation in solving the unbalanced assignment problem

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In a 2019 paper, the authors claim to have developed a modified Hungarian method that performs better than a number of other solution methods for the unbalanced assignment problem (UAP) based on the solution of one UAP instance that has been discussed in the literature. The purpose of this short paper is to demonstrate that the math formulation used in the 2019 paper was not as restrictive as the standard one commonly used in the literature and therefore the comparison is not valid. The commonly used UAP math formulation not only tries to minimize cost, but also tries to level load the jobs onto the machines. The formulation from the 2019 paper allows many jobs to be assigned to a low-cost machine. Hence solutions (not even optimums) to the 2019 formulation can be better than the optimal solution using the standard UAP math formulation. Additionally, it will be shown that the Modified Hungarian method proposed in the 2019 paper does not generate guaranteed optimums to the math formulation used in that paper (let alone the standard UAP formulation). An 8-job and 5-machine assignment problem that appeared in the literature will be used to illustrate the points mentioned above.

1. Introduction

For an assignment problem that has more jobs (n) than machines (m), the assignment problem is considered unbalanced and there are several ways to model an unbalanced assignment problem (UAP). In this paper, two formulations will be discussed [1], [2]. The first formulation will be referred to as the "standard" formulation [3]–[12]. In this case, the emphasis is on level loading the jobs onto the machines so that all machines get about the same number of jobs assigned. The second formulation we will refer to as the RKQ formulation because it is discussed in Rabbani et al. [13]. This formulation ensures that each machine has at least one job assigned to it, but otherwise the objective is to strictly minimize cost. Both of these formulations assume that job processing times and machine availability times are not known.

In Rabbani et al. [13], they introduce a Modified Hungarian method to solve their formulation of the UAP. They then compare the solution they obtain using their UAP formulation and their Modified Hungarian method to the solution obtained by Betts and Vasko [14], who used the "standard" UAP formulation when solving an 8-job and 5-machine assignment problem. Rabbani et al. [13] claimed that their solution with an objective function value of 1470 was better than the Betts and Vasko [14] solution that had an objective function value of 1520. However, the Betts and Vasko

[14] formulation which favors level loading the machines allows only two jobs per machine which is not the case for the Rabbani et al. [13] formulation. Hence the Rabbani et al. [13] comparison is invalid. Additionally, it will be shown below that the solution Rabbani et al. [13] generate for the 8-job and 5-machine assignment problem is not optimum for their UAP formulation.

Hence the purpose of this paper is not to present new theory or methodology for the UAP. The contributions of this paper are simply: (1) to reveal that the claim in the 2019 Rabbani et al. [13] paper that their Modified Hungarian method gives better results than other UAP solution methods is false because their UAP formulation violates the UAP formulation commonly used in the literature, (2) to prove that the Modified Hungarian method of Rabbani et al. [13] does not guarantee optimums.

In the next section, the two UAP formulations will be contrasted in a little more detail illustrated by their solutions to the 8-job and 5-machine UAP from the literature. A few comments will conclude this paper.

2. Method

2.1. Problem statement Unbalanced Assignment Problem Formulations

When there are more jobs than machines and more than one job can be assigned to a machine, a "standard" modeling approach is to make copies of the machines and then simply use the Hungarian method Kuhn [3] to find a guaranteed optimal solution. For example, if there are 19 jobs to be assigned to 5 machines, then in an effort to level load these jobs to machines, no more than 4 = ceiling of 19/5 ($\lceil 19/5 \rceil$) jobs would be assigned to each machine. In this case, the five machines would be "cloned" three times so that there are four machines number one, four machines number two, four machines number three, four machines number four, and four machines number five. To balance this problem, one dummy job would be created and this balanced (20 columns and 20 rows) assignment problem could now be solved using the Hungarian Method [15], [16]. The mathematical formulation for this standard UAP model is given by:

Minimize: $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$

Subject to constraints

$$\sum_{j=1}^{n} X_{ij} \geq 1, \quad i = 1, 2, \dots, m \tag{1}$$

$$\sum_{j=1}^{n} X_{ij} \leq \left\lceil \frac{n}{m} \right\rceil, \quad i = 1, 2, \dots, m \tag{2}$$

$$\sum_{i=1}^{m} X_{ij} = 1, \quad j = 1, 2, \dots, n \tag{3}$$

$X_{ij} = 0 \; or \; 1$

As seen in Equation (1), Equation (2) dan Equation (3), where $X_{ij} = 1$ if the jth job is assigned to machine i and 0 of the jth job is not assigned to machine i. Let C_{ij} be the cost of processing job j on machine i. Equation (1) ensure that each machine gets at least one job assigned to it and Equation (2) make sure that no more than $\left\lceil \frac{n}{m} \right\rceil$ jobs are assigned to each machine. Equation (2) are the level-loading constraints. Equation (3) ensure that each job is assigned to exactly one machine.

The formulation presented in Rabbani et al. [13] ensures that each job is assigned to a machine and that each machine is assigned at least one job. However, this formulation puts no limit on the number of jobs assigned to a machine. In other words, Equation (2) are missing from their formulation. Hence, in order to minimize cost, many jobs can be assigned to one machine. For example, consider a 19-job and 5-machine assignment problem in which the cost to assign jobs to machine #5 is much cheaper than assigning jobs to the other four machines. In this case, in order to

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minimize cost, one job would be assigned to each of machines #1 to #4, and 15 jobs would be assigned to machine #5. Rabbani et al. [13] present a modified Hungarian method to solve the UAP using their formulation. However, as will be shown using the UAP example below, their Modified Hungarian method does not necessarily find the optimum.

2.2. Proposed Method An 8-job and 5-machine UAP example

In order to contrast the two assignment problem formulations mentioned above, we will use an 8-job by 5-machine assignment problem that is discussed and solved in both Betts and Vasko [14] and Rabbani et al. [13] and is given in Table 1 below.

Table 1. Cost Matrix

	Machines				
jobs	M1	M2	M3	M4	M5
J1	300	290	280	290	210
J2	250	310	290	300	200
J3	180	190	300	190	180
J4	320	180	190	240	170
J5	270	210	190	250	160
J6	190	200	220	190	140
J7	220	300	230	180	160
J8	260	190	260	210	180

3. Results and Discussion

Betts and Vasko [14] first balance this problem by cloning the machines once and adding two dummy jobs. The 10 by 10 matrix is solved using the standard Hungarian method which guarantees that the optimum is found. Solution details are provided in Betts and Vasko [14].

In Rabbani et al. [13], their math formulation is used and this problem is solved using their Modified Hungarian method. Solution details are provided in Rabbani et al. [13]. The solutions from these two formulations are given in Table 2.

The solution obtained by Betts and Vasko [14], in which the formulation used requires there to be no more than two jobs assigned to any machine, is optimum for that formulation since the Hungarian method was used to solve it. Rabbani et al. [13] claimed that their solution is better than the Betts and Vasko [14] solution. However, their claim is not valid because the formulations are different and the Betts and Vasko [14] formulation is more restrictive in that it allows at most two jobs assigned to any machine. The Rabbani et al. [13] formulation puts no restrictions on the number of jobs that can be assigned to a machine as long as each machine is assigned at least one job. It is interesting to note that the solution that was obtained by Rabbani et al. [13] using their Modified Hungarian method was not an optimal solution for their formulation. In the last column of Table 2, a proven optimal solution for the Rabbani et al. [13] formulation is given using general-purpose integer programming software (CPLEX). Hence, this proves that their modified Hungarian solution method does not generate guaranteed optimal solutions.

Table 2. Impact of Assignment Problem Formulation on the solution

	Standard formulation and Hungarian solution method	RKQ formulation and their modified Hungarian solution method	Proven optimal solution for the RKQ formulation
Machines	Jobs assigned	Jobs assigned	Jobs assigned
M1	J2, J3	J3	J3
M2	J4, J8	J4, J8	J8
M3	J5	J5	J4
M4	J7	J7	J7
M5	J1, J6	J1, J2, J6	J1, J2, J5, J6
Total Cost	1520	1470	1450

4. Conclusion

In this short paper, we demonstrated that the formulation used to model an unbalanced assignment problem can impact the optimal solutions generated. In particular, Rabbani et al. [13] failed to take into consideration that their UAB formulation was less restrictive than the standard one used in the operations research literature. Hence the claim by them [13] that their Modified Hungarian method obtained better results than solution approaches that used the standard UAP formulation was invalid. Furthermore, it was also shown that the Modified Hungarian solution method used to solve assignment problems based on the less restrictive formulation given in Rabbani et al. [13] does not find guaranteed optimal solutions. If an unbalanced assignment problem needs to be solved in an industrial application, we suggest using the standard UAP formulation so that the solution will level-load the machines.

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References

[1] V. Amarnadh and N. R. Moparthi, “Range control-based class imbalance and optimized granular elastic net regression feature selection for credit risk assessment,” *Knowl. Inf. Syst.*, 2024, doi: 10.1007/s10115-024-02103-9.

[2] J. Wang, W. Wang, X. Hu, L. Qiu, and H. Zang, “Black-winged kite algorithm: a nature-inspired meta-heuristic for solving benchmark functions and engineering problems,” *Artificial Intelligence Review*. Springer, 2024, doi: 10.1007/s10462-024-10723-4.

[3] H. W. Kuhn, “The Hungarian method for the assignment problem,” *Nav. Res. Logist. Q.*, vol.

2, no. 1–2, pp. 83–97, Mar. 1955, doi: 10.1002/nav.3800020109.

[4] F. S. Hillier, "Lieberman and J. Gerald,'Introduction to Operations Research.'" McGraw-Hill, New York, 2010.

[5] M. Ramesh, "Lexi-Search Approach to Some Combinatorial Programming Problem," *University of Hyderabad, India*. igmlnet.uohyd.ac.in, 1997, [Online]. Available: https://igmlnet.uohyd.ac.in/docs/hi-res/hcu_images/TH2459.pdf.

[6] W. L. Winston and J. B. Goldberg, "Operations research: applications and algorithms (Vol. 3)," *Belmont: Thomson Brooks/Cole*. 2004.

[7] V. Yadaiah and V. V. Haragopal, "A New Approach of Solving Single Objective Unbalanced Assignment Problem," *Am. J. Oper. Res.*, vol. 06, no. 01, pp. 81–89, 2016, doi: 10.4236/ajor.2016.61011.

[8] S. Dhouib, "An intelligent assignment problem using novel heuristic: The dhouib-matrix-ap1 (dm-ap1): Novel method for assignment problem," *International Journal of Intelligent Systems and Application in Engineering*. 2022. doi: 10.18201/ijisae.2022.277

[9] S. Dhouib, "Novel optimization method for unbalanced assignment problems with multiple jobs: The Dhouib-Matrix-AP2," *Intell. Syst. with Appl.*, vol. 17, p. 200179, Feb. 2023, doi: 10.1016/j.iswa.2023.200179.

[10] A. Kumar, "A modified method for solving the unbalanced assignment problems," *Appl. Math. Comput.*, 2006, [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0096300305007721.

[11] J. Majumdar and A. K. Bhunia, "An alternative approach for unbalanced assignment problem via genetic algorithm," *Appl. Math. Comput.*, vol. 218, no. 12, pp. 6934–6941, Feb. 2012, doi: 10.1016/j.amc.2011.12.070.

[12] R. K. Mondal, P. Ray, E. Nandi, B. Biswas, and M. K. Sanyal, "Load balancing of unbalanced assignment problem with hungarian method," *Int. J. Ambient Computing and Intelligence*, 2019. doi: 10.4018/IJACI.2019010103.

[13] Q. Rabbani, A. Khan, and A. Quddoos, "Modified Hungarian method for unbalanced assignment problem with multiple jobs," *Appl. Math. Comput.*, vol. 361, pp. 493–498, Nov. 2019, doi: 10.1016/j.amc.2019.05.041.

[14] N. Betts and F. J. Vasko, "Solving the Unbalanced Assignment Problem: Simpler Is Better," *Am. J. Oper. Res.*, vol. 06, no. 04, pp. 296–299, 2016, doi: 10.4236/ajor.2016.64028.

[15] M. Khashei, M. Ahmadi, and F. Chahkoutahi, "A mean weighted squared error-based neural classifier for intelligent pattern recognition in smart grids," *International Journal of Electrical Power & Energy Systems*. Elsevier, 2025, [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0142061525005204.

[16] F. Fouad, A. E. H. Kassam, and F. F. Al-Obaidi, "A new heuristic method for solving unbalanced multi-objective assignment problem," *Eng. Res. Express*, 2024, doi: 10.1088/2631-8695/ad9888.