**Bulletin of Applied Mathematics and Mathematics Education**
Volume 2, Number 2, October 2022, pp. 59-64
https://dx.doi.org/10.12928/bamme.v2i2.7072
e-ISSN 2776-1029, p-ISSN 2776-1002

# On Conformable, Riemann-Liouville, and Caputo fractional derivatives

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ARTICLE INFO

**Article History**
Received 17 December 2022
Revised 29 December 2022
Accepted 29 December 2022

**Keywords**
Caputo fractional derivative,
Conformable fractional derivative,
Fractional ordinary differential equations,
Riemann-Liouville fractional derivative,
Solutions.

**How to cite this article:**
Guswanto, B. H., Andini, L. R., & Triyani. (2022). On Conformable, Riemann-Liouville, and Caputo fractional derivatives. *Bulletin of Applied Mathematics and Mathematics Education*, *2*(2), 59-64.

ABSTRACT

This article compares conformable fractional Derivative with Riemann-Liouville and Caputo fractional derivative by comparing solutions to fractional ordinary differential equations involving the three fractional derivatives via the numerical simulations of the solutions. The result shows that conformable fractional derivative can be used as an alternative to Riemann-Liouville and Caputo fractional derivative for order $\alpha$ with $1/2 < \alpha < 1$.



## Introduction

Recently, the study of fractional derivative notion has grown fast not only in theory but also in applications. It is motivated by many real phenomena that cannot be solved by standard or usual derivative. For example, slow diffusion or sub-diffusion process cannot be modeled well by usual diffusion or heat equation involving first derivative with respect to time t. In subdiffusion process, the mean square displacement of a particle moving in this process is proportional to $t^\alpha$ with $0 < \alpha < 1$. Of course, sub-diffusion process cannot be modeled by usual diffusion equation describing usual diffusion process in which the mean square displacement of a particle moving in this process is proportional linearly to $t$. Such phenomena can be found in various studies such as Adams and Gelhar (1992), Berkowitz et al. (2006), Hatano and Hatano (1988), and Laffaldano et al. (2005). The other examples are viscoelastic materials, as reported in Podlubny (1999). Viscoelastic materials have mechanical characteristics intermediate between those of viscous liquid and those of elastic solid. An elastic material satisfies Hooke law i.e.

$$\sigma(t) = E\varepsilon(t) = E\frac{d^0}{dt^0}\varepsilon(t)$$

where $\sigma$ and $\varepsilon$ are stress and strain of the material at $t > 0$, respectively, and $E$ is a material elasticity modulus constant. A viscous material satisfies Newton law i.e.

$$\sigma(t) = V\frac{d}{dt}\varepsilon(t) = V\frac{d^1}{dt^1}\varepsilon(t)$$

where $V$ is a material viscosity constant. How about viscoelastic materials such as synthetic polymers, aluminium metal, and paste? What law is satisfied by the materials? Viscoelastic materials satisfy a law

$$\sigma(t) = C\frac{d^\alpha}{dt^\alpha}\varepsilon(t)$$

where $C$ is a material viscoelasticity constant and $d^\alpha/dt^\alpha$ is a fractional derivative operator of orde $\alpha$ with $0 < \alpha < 1$. The operator $d^\alpha/dt^\alpha$ describes a state "between" the operator $d^0/dt^0$ and the operator $d^1/dt^1$.

There are some types of fractional derivative operators. Some well-known of fractional derivative operators are Riemann-Liouville and Caputo fractional derivatives. Nevertheless, both fractional derivative operators does not satisfy some properties of usual derivative such as multiplication rule, division rule, chain rule, Rolle theorem, and mean value theorem. Khalil et al. [6] then gives new definition of fractional derivative satisfying the properties of usual derivative which are not satisfied by Riemann-Liouville and Caputo fractional derivative. This fractional derivative is called conformable fractional derivative. Abdeljawad (2015) then develops the conformable fractional derivative and gives more comprehensively its properties. Besides, conformable fractional derivative is defined simpler than the definition of Riemann-Liouvile and Caputo fractional derivatives. Differently from Riemann-Liouville and Caputo which are nonlocal operators, the conformable fractional derivative is a local operator since it is defined similar to the definition of usual derivative.

This paper discusses the comparation of conformable, Riemann-Liouville, and Caputo fractional derivatives of order $\alpha$ with $0 < \alpha < 1$. To compare them, we simulate numerically the solutions to fractional ordinary differential equations involving the three fractional derivatives.

This paper is composed of five sections. The second section discusses briefly Riemann-Liouville, Caputo, and conformable fractional derivatives. In the third section, the solutions to fractional ordinary differential equations involving Riemann-Liouville, Caputo, and conformable are given. Furthermore, comparation of numerical simulation for the solutions is also given. A conclusion is provided in the last section.

## Riemann-Liouville, Caputo, and Conformable fractional derivatives

This section discusses briefly Riemann-Liouville, Caputo, and Conformable fractional derivatives. One can refer to Podlubny (1999) and Kilbas et al. (2006) for more detail concerning Riemann-Liouville and Caputo fractional derivatives, and Abdeljawad (2015) and Khalil et al. (2014) for more detail regarding conformable fractional derivative.

We first define fractional integral $I_t^\alpha$ of order $\alpha > 0$ for a integrable function $f: (0, \infty) \to \mathbb{R}$ as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad t > 0.$$

We suppose $n \in \mathbb{N}$. Riemann-Liouville fractional derivative ${}^R D_t^\alpha$ of order $\alpha \in (n-1, n]$ for $f$


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is then defined as

$$^{R}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha+1}f(s)ds\,,\qquad t>0.$$

Next, Caputo fractional derivative $^{C}D_t^{\alpha}$ of order $\alpha\in(n-1,n]$ for $f$ is defined as

$$^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha+1}\frac{d^n}{ds^n}f(s)ds\,,\qquad t>0.$$

Relationship between Riemann-Liouville and Caputo fractional derivatives is given by the identity

$$^{C}D_t^{\alpha}f(t)=\,^{R}D_t^{\alpha}f(t)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0)\,,\qquad t>0.$$

Some properties satisfied by usual derivative such as multiplication rule, division rule, chain rule, Rolle theorem, and mean value theorem are not satisfied by both fractional derivatives.

We now provide the definition of conformable fractional derivative $T_t^{\alpha}$ of order $\alpha\in(n-1,n]$ for $(\lceil\alpha\rceil-1)$-times differentiable function $f$ at $t>0$ as

$$T_t^{\alpha}f(t)=\lim_{\varepsilon\to0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}$$

where $\lceil\alpha\rceil$ denotes the least integer that is greater than or equal to $\alpha$. It follows that

$$T_t^{\alpha}f(t)=t^{\lceil\alpha\rceil-\alpha}f^{(\lceil\alpha\rceil)}(t).$$

## Fractional ordinary differential equations

This section discusses solutions to fractional Ordinary Differential Equations using Riemann-Liouville, Caputo, and conformable fractional Derivatives and then their numerical simulations. For more detail concerning these, one can refer to Abdeljawad (2015), Khalil et al. (2014), Podlubny (1999), and Kilbas (2006).

Solutions

The initial value problem of fractional ordinary differential equations with Riemann-Liouville fractional derivative, for $0<\alpha<1$,

$$^{R}D_t^{\alpha}y(t)=\lambda y(t),\qquad t>0,$$

with the initial condition

$$I_t^{1-\alpha}y(t)|_{t=0}=y_0,$$

has the solution

$$y(t)=y_0t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler Function defined by

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\qquad \alpha,\beta>0.$$

The initial value problem of fractional ordinary differential equations with Caputo fractional derivative, for $0<\alpha<1$,

$$^{C}D_t^{\alpha}y(t)=\lambda y(t),\qquad t>0,$$

with the initial condition

$$y(t)|_{t=0} = y_0,$$

has the solution

$$y(t) = y_0 E_{\alpha,1}(\lambda t^{\alpha}).$$

The initial value problem of fractional ordinary differential equations with conformable fractional derivative, for $0 < \alpha < 1$,

$$T_t^{\alpha} y(t) = \lambda y(t), \qquad t > 0,$$

with the initial condition

$$y(t)|_{t=0} = y_0,$$

has the solution

$$y(t) = y_0 e^{\lambda \frac{t^{\alpha}}{\alpha}}.$$

## Numerical simulations

The solutions to the fractional ordinary differential equations is simulated numerically with $y_0 = 1$ and $0 \leq t \leq 10$. The graphs of the solutions to the fractional ordinary differential equations with Riemann-Liouville, Caputo, and conformable fractional derivative are colored by red, yellow, and green, respectively.

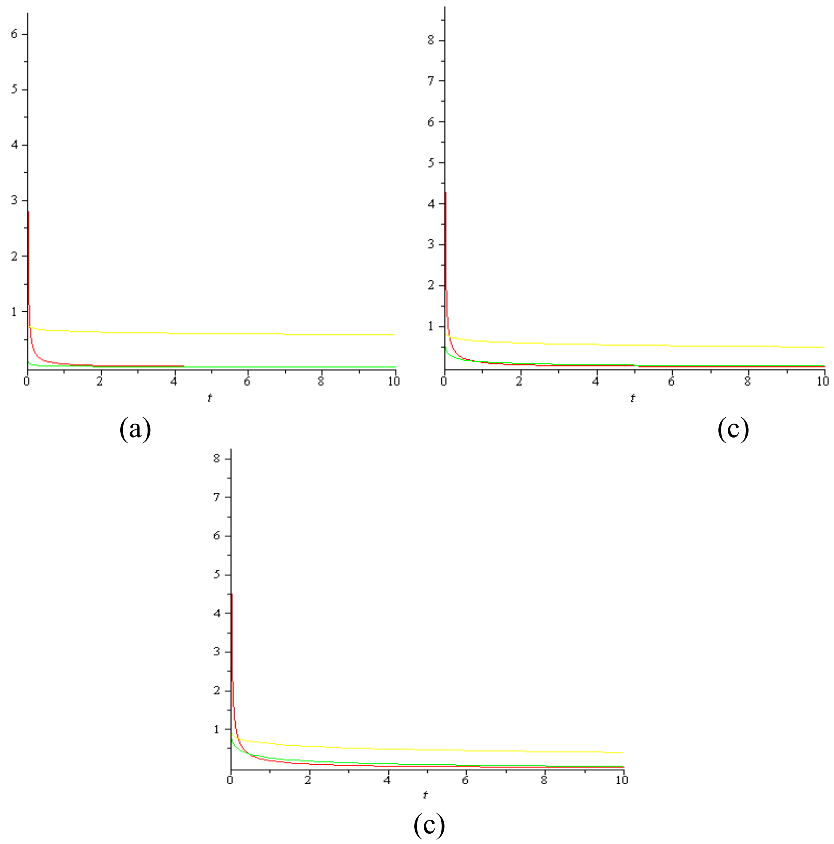


**Figure 1.** The graphs of the solutions for (a) $\alpha = 1/8$ (b) $\alpha = 1/4$ (c) $\alpha = 3/8$

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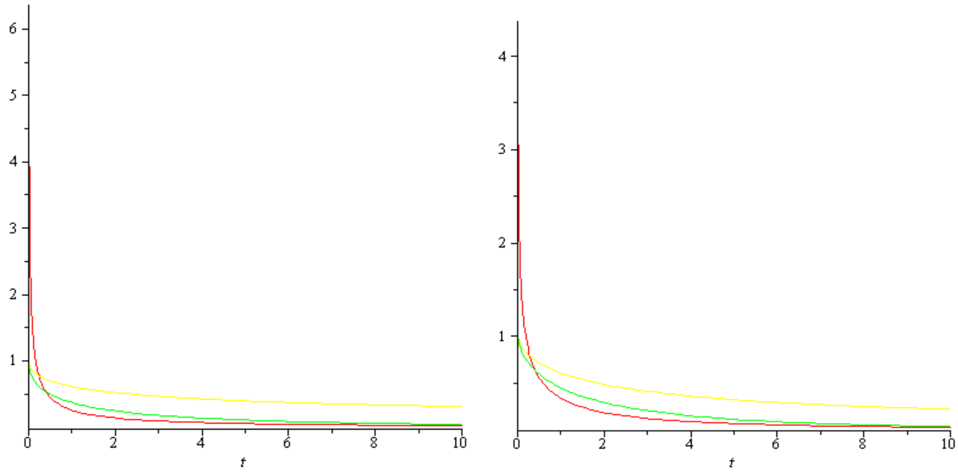
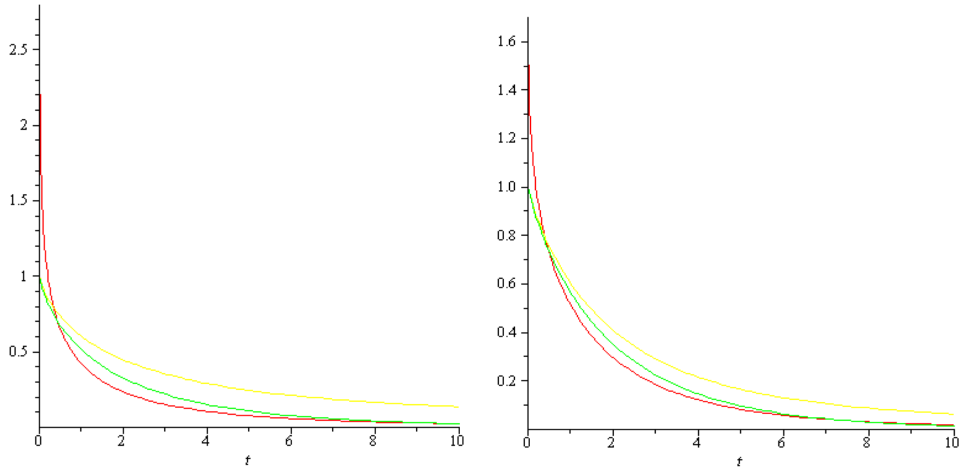



(a) (b)

**Figure 2. .** The graphs of the solutions for (a) $\alpha = 1/2$ (b) $\alpha = 5/8$




(a) (b)

**Figure 3.** The graphs of the solutions for (a) $\alpha = 3/4$ (b) $\alpha = 7/8$

Figure 1, Figure 2, and Figure 3 show us that the graphs of the solutions with Riemann-Liouville and conformable fractional derivatives are sufficiently close for $0 < \alpha < 1$. Moreover, based on Figures 1 and 2(a), there is a wide gap between the graph of the solution with Caputo fractional derivative and the other two for $0 < \alpha \leq 1/2$. The graphs of the solutions with the three fractional derivatives are sufficiently close for $1/2 < \alpha < 1$ as shown by Figures 2(b) and 3. If $\alpha$ gets closer to 1 then the graphs of the solution with the three fractional derivatives are closer each other, and, for $\alpha = 1$, the graphs are the same.

## Conclusion

The numerical simulations of the solutions to the initial value problems of the fractional ordinary differential equations show us that the graphs of solutions with Riemann-Liouville and conformable fractional derivatives are sufficiently close for $0 < \alpha < 1$. The graphs of the solutions with the three fractional derivatives are sufficiently close for $1/2 < \alpha < 1$. Thus, conformable fractional derivative can be used as an alternative to Riemann-Liouville and Caputo fractional derivatives for $1/2 < \alpha < 1$.

The advantage of the use of conformable fractional derivative is on its definition which is simpler than those of Riemann-Liouville and Caputo fractional derivatives. The other advantage is that some properties of usual derivative which are not satisfied by Riemann-Liouville and Caputo are in fact satisfied by conformable fractional derivative.

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